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IMPACT ON AN ELASTICALLY CONNECTED  
DOUBLE BEAM SYSTEM

by

J. M. Seelig

and

W. H. Hoppmann II

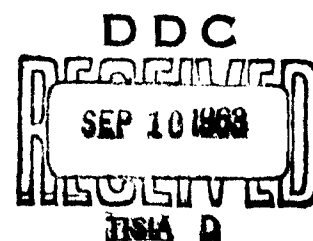
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Troy, New York

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# IMPACT ON AN ELASTICALLY CONNECTED

## DOUBLE BEAM SYSTEM

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and

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### ABSTRACT

The report presents the development and solution of the differential equations of motion of an elastically connected double beam system subjected to an impulsive load.

In addition to the theory there is presented a description of impact experiments on double beams. Theoretically and experimentally determined strains as functions of time are compared in the form of curves. The agreement is remarkably good for the type of load used.

### NOMENCLATURE

E	=	Young's modulus of elasticity
I	=	moment of inertia of cross section of beam about neutral axis
$\rho$	=	mass per unit volume
k	=	modulus of elastic connectors

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$w$	=	deflection of beam
$\omega_n$	=	circular frequency of $n^{\text{th}}$ mode of vibration
$l$	=	length of beam
$I_M$	=	mass moment of inertia of rigid beam
$t$	=	time
$\tau$	=	contact time of impulsive load
$\sigma$	=	stress
$c$	=	distance from neutral axis to outer fiber of beam
$F(x,t)$	=	impulsive load acting at any point $x$ at any time $t$
$A$	=	area of cross section of beam
$\gamma$	=	density
$g$	=	acceleration due to gravity
$\phi_n$	=	mode shape associated with $n^{\text{th}}$ mode of vibration
$x$	=	space variable coordinate reference at end of bar
$X(x)$	=	function of space variable only
$\omega$	=	$\frac{\pi}{\tau}$
$a_0$	=	impact position
$n$	=	number of $n^{\text{th}}$ mode of vibrations

## INTRODUCTION

Increasingly in the last hundred years, considerable study has been devoted to the problem of the impact loading of structures. Since World War II an impressive amount of theory and experiment has been developed and knowledge in the field has been effectively extended. Many of the results are

described in detail in a book on impact by Werner Goldsmith [1]<sup>3</sup>. Another recent study of the effect of impact on structures is provided by Hoppmann in the Shock and Vibration Handbook, edited by Harris and Crede [2]. An exceptionally fine experimental investigation of impact for a single beam is given in a paper by Hans-Heinrich Emschermann and Karl Rühl [3].

Probably the most extensively studied structural element from the standpoint of transverse dynamic loading is the single beam. The Bernoulli-Euler theory for dynamic loading of a beam has been shown to be very useful if the restrictive assumptions on which the theory is based are observed. It has been demonstrated experimentally by Hoppmann [4] that such a theory is quite satisfactory even in the case of a multi-span beam.

An important technological extension of the concept of the single beam on various kinds of end supports is that of the beam on an elastic foundation. A useful development of the idea of beams supported on spring-like foundations is provided in an excellent treatise on the subject by Hetényi [5]. However, all of the treatment there is limited to the case of static loading. The theory of the transverse dynamic loading of a beam on spring-like elastic foundation was worked out from the standpoint of the Bernoulli-Euler theory by Hoppmann [6].

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<sup>3</sup>Numbers in brackets designate References at the end of report.



Subsequently, Wenk [7] studied the response of a thin closed ring on elastic foundation both theoretically and experimentally. The results are very encouraging.

An application of beam theory to the case of the vibration of double beams which are elastically coupled has been attempted by Dublin and Friedrich [8] and Osborne [9].

The problem of the impact loading of two beams which are elastically connected is of technological interest and has been extensively studied in the present research. In particular, experiments have been conducted in which two free beams elastically connected and poised in space have been subjected to impulsive loads. These experimental results have been compared with the theory which was developed,

#### DIFFERENTIAL EQUATIONS OF MOTION OF A FREE DOUBLE BEAM

The differential equations of motion of an elastically connected free double beam system under impulsive load are according to the Bernoulli-Euler beam theory:

$$EI \frac{\partial^4 w_1}{\partial x^4} = -k(w_1 - w_2) - \frac{A\gamma}{g} \frac{\partial^2 w_1}{\partial t^2} + F(x, t) \quad (1)$$

$$EI \frac{\partial^4 w_2}{\partial x^4} = k(w_1 - w_2) - \frac{A\gamma}{g} \frac{\partial^2 w_2}{\partial t^2} \quad (2)$$

From Eq. (2), solve for  $w_1$ .

$$w_1 = \frac{EI}{k} \frac{\partial^4 w_2}{\partial x^4} + \frac{AY}{gk} \frac{\partial^2 w_2}{\partial t^2} + w_2 \quad (3)$$

Substituting Eq. (3) into Eq. (1) gives a partial differential equation in  $w_2$  alone.

$$a^2 \frac{\partial^8 w_2}{\partial x^8} + 2ac \frac{\partial^6 w_2}{\partial x^4 \partial t^2} + 2a \frac{\partial^4 w_2}{\partial x^4} + c^2 \frac{\partial^4 w_2}{\partial t^4} + 2c \frac{\partial^2 w_2}{\partial t^2} = \frac{F(x,t)}{k} \quad (4)$$

$$\text{where } a = \frac{EI}{k} \text{ and } c = \frac{AY}{g}.$$

With  $F(x,t) = 0$ , the homogeneous partial differential equation (4) has the solution

$$w_2 = \alpha_1 x t + \alpha_2 x + \alpha_3 t + \alpha_4 + \sum_{n=1}^{\infty} \Phi_n (A_n \cos \omega_n t + B_n \sin \omega_n t) \\ + \alpha_5 \sin \omega_0 t + \alpha_6 \cos \omega_0 t + \alpha_7 x \sin \omega_0 t + \alpha_8 x \cos \omega_0 t \quad (5)$$

where

$$\Phi_n = \frac{\cosh k_n x + \cos k_n x}{\cosh k_n l - \cos k_n l} - \frac{\sinh k_n x + \sin k_n x}{\sinh k_n l - \sin k_n l} \quad (6)$$

and

$$\cosh k_n l \cos k_n l = 1, \quad \omega_0^2 = \frac{2k}{\rho}. \quad (7)$$

In Eq. (5), the terms involving the  $\alpha$ 's are from rigid body motions. If each of the two beams is considered to be rigid and furthermore both beams are elastically connected, the differential equations of motion for the system under impulsive load  $P \sin \omega t$  applied at  $c_0$  from the center line are:

$$\rho l \ddot{w}_1 = P \sin \omega t - kl(w_1 - w_2) \quad (8)$$

$$\rho l \ddot{w}_2 = kl(w_1 - w_2) \quad (9)$$

$$I_M \ddot{\theta}_1 = Pc_0 \sin \omega t - \frac{kl^3}{12} (\theta_1 - \theta_2) \quad (10)$$

$$I_M \ddot{\theta}_2 = \frac{kl^3}{12} (\theta_1 - \theta_2) \quad (11)$$

Solving these equations, the coefficients  $\alpha_1$  through  $\alpha_8$  are determined by comparison with Eq. (5) assuming the vibratory portion to be small.

The  $\alpha$ 's are found to be:

$$\begin{aligned} \alpha_1 &= a' & \alpha_2 &= 0 & \alpha_3 &= a - \frac{a'}{2} l & \alpha_4 &= 0 \\ \alpha_5 &= d - \frac{d'}{2} l & \alpha_6 &= 0 & \alpha_7 &= d' & \alpha_8 &= 0 \end{aligned} \quad (12)$$

where

$$\begin{aligned}
a &= \frac{P}{2\omega\rho l} & d &= P \frac{\omega}{\omega_0} \left[ \frac{\frac{1}{\omega_0^2}}{2\rho l(1 - \frac{\omega^2}{\omega_0^2})} \right] \\
a' &= \frac{Pc_0}{2\omega I_M} & d' &= Pc_0 \frac{\omega}{\omega_0} \left[ \frac{\frac{1}{\omega_0^2}}{2I_M(1 - \frac{\omega^2}{\omega_0^2})} \right]
\end{aligned} \tag{13}$$

#### DISPLACEMENTS OF BEAMS CAUSED BY CONCENTRATED IMPULSIVE LOAD

In order to obtain the particular solution of Eq. (4), assume the impact force to be expressible as follows:

$$\frac{F(x,t)}{k} = (c_3 + c_4 x + \sum_{n=1}^{\infty} p_n \Phi_n) \sin \omega t \tag{14}$$

where

$$(c_3 + c_4 x + \sum_{n=1}^{\infty} p_n \Phi_n) = \frac{P}{k} \tag{15}$$

The particular solution of Eq. (4) is, therefore,

$$w_P = (c_1 + c_2 x + \sum_{n=1}^{\infty} w_n \Phi_n) \sin \omega t \tag{16}$$

Again considering rigid body terms from the solutions of Eqs. (8) through (11), the coefficients  $c_1$  and  $c_2$  in Eq. (16) are evaluated as

$$c_1 = \left[ \frac{-\frac{1}{\omega^2}}{2\rho l} + \frac{-\frac{1}{\omega_0^2}}{2\rho l(1 - \frac{\omega^2}{\omega_0^2})} \right] P - \left[ \frac{-\frac{1}{\omega^2}}{2I_M} + \frac{-\frac{1}{\omega_0^2}}{2I_M(1 - \frac{\omega^2}{\omega_0^2})} \right] P c_0 \frac{l}{2} \quad (17)$$

and

$$c_2 = \frac{-\frac{1}{\omega^2}}{2I_M} - \frac{\frac{1}{\omega_0^2}}{2I_M(1 - \frac{\omega^2}{\omega_0^2})} \quad (18)$$

Substitute Eqs. (14) and (16) into Eq. (4)

$$\begin{aligned} a^2 \sum_{n=1}^{\infty} w_n \frac{d^8 \phi_n}{dx^8} - 2ac \omega^2 \sum_{n=1}^{\infty} w_n \frac{d^4 \phi_n}{dx^4} + 2a \sum_{n=1}^{\infty} w_n \frac{d^4 \phi_n}{dx^4} + \\ + \omega^4 c^2 \left[ c_1 + c_2 x + \sum_{n=1}^{\infty} w_n \phi_n \right] - 2c \omega^2 \left[ c_1 + c_2 x + \sum_{n=1}^{\infty} w_n \phi_n \right] = \\ = c_3 + c_4 x + \sum_{n=1}^{\infty} p_n \phi_n \end{aligned} \quad (19)$$

Equating like terms in Eq. (19), allows  $c_3$  and  $c_4$  to be evaluated

$$c_3 = c_1 (\omega^4 c^2 - 2c \omega^2) \quad (20)$$

$$c_4 = c_2 (\omega^4 c^2 - 2c \omega^2) \quad (21)$$

Equation (19) can now be simplified to

$$a^2 \sum_{n=1}^{\infty} w_n \frac{d^8 \phi_n}{dx^8} - (2ac \omega^2 - 2a) \sum_{n=1}^{\infty} w_n \frac{d^4 \phi_n}{dx^4} + (\omega^4 c^2 - 2c \omega^2) \sum_{n=1}^{\infty} w_n \phi_n = \sum_{n=1}^{\infty} p_n \phi_n \quad (22)$$

From free vibrations it is known that

$$a^2 \sum_{n=1}^{\infty} w_n \frac{d^8 \phi_n}{dx^8} - (2ac \omega_n^2 - 2a) \sum_{n=1}^{\infty} w_n \frac{d^4 \phi_n}{dx^4} + (\omega_n^4 c^2 - 2c \omega_n^2) \sum_{n=1}^{\infty} w_n \phi_n = 0 \quad (23)$$

By combining Eqs. (22) and (23) and simplifying,  $w_n$  can be determined.

$$w_n = \frac{p_n \phi_n}{2ac(\omega_n^2 - \omega^2) \frac{d^4 \phi_n}{dx^4} - [c^2(\omega_n^4 - \omega^4) - 2c(\omega_n^2 - \omega^2)] \phi_n} \quad (24)$$

The shape function  $\phi_n$  is such that

$$\frac{d^4 \phi_n}{dx^4} = k_n^4 \phi_n \quad (25)$$

Equation (24) therefore becomes

$$w_n = \frac{p_n}{2ac k_n^4 (\omega_n^2 - \omega^2) - [c^2 (\omega_n^4 - \omega^4) - 2c (\omega_n^2 - \omega^2)]} \quad (26)$$

The general solution of Eq. (4) is

$$\begin{aligned} w_2 = & (\alpha_1 x + \alpha_3) t + (\alpha_5 + \alpha_7 x) \sin \omega_0 t + \\ & + \sum_{n=1}^{\infty} \Phi_n [B_n \sin \omega_n t + A_n \cos \omega_n t] + (c_1 + c_2 x) \sin \omega t + \\ & + \sum_{n=1}^{\infty} w_n \Phi_n \sin \omega t \end{aligned} \quad (27)$$

Assuming that the system is initially at rest,  $A_n$  and  $B_n$  can be determined as:

$$A_n = 0 \quad B_n = - \frac{\omega}{\omega_n} w_n \quad (28)$$

Equation (27) becomes

$$\begin{aligned} w_2 = & (\alpha_1 x + \alpha_3) t + (\alpha_5 + \alpha_7 x) \sin \omega_0 t + (c_1 + c_2 x) \sin \omega t + \\ & + \sum_{n=1}^{\infty} \Phi_n B_n \sin \omega_n t + \sum_{n=1}^{\infty} w_n \Phi_n \sin \omega t \end{aligned} \quad (29)$$

From Eq. (3),  $w_1$  can be determined, completing the solution of the system of differential equations.

$$\begin{aligned}
w_1 = & (\alpha_1 x + \alpha_3)t + \left[1 - \frac{A\gamma}{gk} \omega_0^2\right](\alpha_5 + \alpha_7 x) \sin \omega_0 t + \\
& + (c_1 + c_2 x) \left[1 - \frac{A\gamma}{gk} \omega^2\right] \sin \omega t + \\
& + \sum_{n=1}^{\infty} \left[ \frac{EI}{k} k_n^4 - \frac{A\gamma}{gk} \omega_n^2 + 1 \right] B_n \Phi_n \sin \omega_n t + \\
& + \sum_{n=1}^{\infty} \left[ k_n^4 \frac{EI}{k} - \frac{A\gamma}{gk} \omega^2 + 1 \right] w_n \Phi_n \sin \omega t \quad (30)
\end{aligned}$$

The only undetermined quantity is  $p_n$ . Using relationship (15), rearranging terms and multiplying by an arbitrary  $\Phi_m$

$$\left(\frac{P}{k} - c_3 - c_4 x\right) \Phi_m = \sum_{n=1}^{\infty} p_n \Phi_m \Phi_n \quad (31)$$

Using the orthogonality relationships

$$\begin{aligned}
\int_0^l \Phi_n \Phi_m dx &= 0 & n \neq m \\
\int_0^l \Phi_n^2 dx &= l \frac{\cos^2 k_n l}{\sin^4 k_n l} & n = m
\end{aligned} \quad (32)$$

Eq. (31) can be solved for  $p_n$ .

$$p_n = \frac{\int_0^l \left(\frac{P}{k} - c_3 - c_4 x\right) \Phi_n dx}{l \frac{\cos^2 k_n l}{\sin^4 k_n l}} \quad (33)$$



The values of  $p_n$  from Eq. (33) are evaluated as follows. Assume  $P$  to be a distributed force of magnitude per unit length  $\frac{P}{c}$  distributed over a small length of beam,  $c$ . As  $c \rightarrow 0$ , in the limit, the product  $\frac{P}{c} \times c$  goes to  $P$ , the concentration. Therefore  $P$  may be written as

$$\begin{aligned}
 P &= 0 & 0 \leq x < a_0 - \frac{c}{2} \\
 P &= \frac{P}{c} & a_0 - \frac{c}{2} \leq x \leq a_0 + \frac{c}{2} \\
 P &= 0 & a_0 + \frac{c}{2} < x \leq l
 \end{aligned} \tag{34}$$

where  $a_0$  is the distance to the point of application of  $P$ .

Equation (33) then becomes

$$p_n = \frac{\lim_{c \rightarrow 0} \int_{a_0 - \frac{c}{2}}^{a_0 + \frac{c}{2}} \frac{P}{kc} \phi_n dx - \int_0^l c_3 \phi_n dx - \int_0^l c_4 x \phi_n dx}{l \frac{\cos^2 k_n l}{\sin^4 k_n l}} \tag{35}$$

which is evaluated using Eq. (6).

$$p_n = \frac{\frac{P}{k} \left[ \frac{\cosh k_n a_0 + \cos k_n a_0}{\cosh k_n l - \cos k_n l} - \frac{\sinh k_n a_0 + \sin k_n a_0}{\sinh k_n l - \sin k_n l} \right]}{l \frac{\cos^2 k_n l}{\sin^4 k_n l}} - \frac{\frac{c_3'}{k_n} \left[ \frac{\sinh k_n l + \sin k_n l}{\cosh k_n l - \cos k_n l} - \frac{\cosh k_n l - \cos k_n l}{\sinh k_n l - \sin k_n l} \right]}{l \frac{\cos^2 k_n l}{\sin^4 k_n l}} \quad (36)$$

where  $c_3' = c_3 + c_4 l$ .

The displacements  $w_1$  and  $w_2$ , during contact, are now completely determined.

For the time after contact, when  $F(x, t) = 0$ , the solution of the differential equation (4) is given by (5) for  $t \geq \tau$ . Let us call this displacement  $w_2'$ . At time  $t = \tau$  the following conditions must hold.

$$w_2(\tau) = w_2'(\tau) \quad \text{and} \quad \dot{w}_2(\tau) = \dot{w}_2'(\tau) \quad (37)$$

Substituting Eqs. (29) and (4) into relationships (37) will enable us to determine the displacement of the system after contact. Following this procedure

$$\begin{aligned}
w_2' = & a_1 x t + a_2 x + a_3 t + a_4 + a_5 \sin \omega_0 \tau + a_6 \cos \omega_0 \tau \\
& + a_7 x \sin \omega_0 \tau + a_8 x \cos \omega_0 \tau \\
& - \sum_{n=1}^{\infty} \frac{\omega}{\omega_n} w_n \phi_n \left[ 2 \cos \frac{\omega_n \tau}{2} \sin \omega_n \left( t - \frac{\tau}{2} \right) \right] \quad (38)
\end{aligned}$$

$$\begin{aligned}
w_1' = & a_1 x t + a_2 x + a_3 t + a_4 \\
& + \left[ 1 - \frac{A\gamma}{gk} \omega_0^2 \right] \left\{ (a_5 + a_7 x) \sin \omega_0 t + (a_6 + a_8 x) \cos \omega_0 t \right\} \\
& + \sum_{n=1}^{\infty} \frac{\omega}{\omega_n} w_n \phi_n \left[ \frac{EI}{k} k_n^4 - \frac{A\gamma}{gk} \omega_n^2 + 1 \right] \left[ 2 \cos \frac{\omega_n \tau}{2} \sin \omega_n \left( t - \frac{\tau}{2} \right) \right] \quad (39)
\end{aligned}$$

The  $a_1$  through  $a_8$  can be evaluated using Eqs. (37), but as we are concerned with strains, when the expression for displacement are differentiated twice with respect to the space variable  $x$ , the rigid body terms will drop out and therefore these terms will not be evaluated.

#### STRAINS IN THE BEAMS

Subject to the assumption of elementary beam theory the strains in a beam may be taken as:

$$\epsilon = c \frac{\partial^2 w}{\partial x^2} \quad (40)$$

Using expression (40), the strains in the beam can be determined.

From Eqs. (29) and (30), the strains during contact are

$$\begin{aligned} \epsilon_1 = c \left\{ \sum_{n=1}^{\infty} \left[ \frac{EI}{k} k_n^4 - \frac{A\gamma}{gk} \omega_n^2 + 1 \right] \frac{d^2 \Phi_n}{dx^2} B_n \sin \omega_n t + \right. \\ \left. + \left[ \frac{EI}{k} k_n^4 - \frac{A\gamma}{gk} \omega_n^2 + 1 \right] w_n \frac{d^2 \Phi_n}{dx^2} \sin \omega t \right\} \quad (41) \end{aligned}$$

$$\epsilon_2 = c \left\{ \sum_{n=1}^{\infty} w_n \frac{d^2 \Phi_n}{dx^2} \left[ \sin \omega t - \frac{\omega}{\omega_n} \sin \omega_n t \right] \right\} \quad (42)$$

$$0 \leq t \leq \tau$$

From Eqs. (38) and (39) the strains after contact are

$$\begin{aligned} \epsilon_1 = -c \left\{ \sum_{n=1}^{\infty} \left[ \frac{EI}{k} k_n^4 - \frac{A\gamma}{gk} \omega_n^2 + 1 \right] \frac{\omega}{\omega_n} w_n \frac{d^2 \Phi_n}{dx^2} \cdot \right. \\ \left. \cdot \left[ 2 \cos \frac{\omega_n \tau}{2} \sin \omega_n \left( t - \frac{\tau}{2} \right) \right] \right\} \quad (43) \end{aligned}$$

$$\epsilon_2 = -c \left\{ \sum_{n=1}^{\infty} \frac{\omega}{\omega_n} w_n \frac{d^2 \Phi_n}{dx^2} \left[ 2 \cos \frac{\omega_n \tau}{2} \sin \omega_n \left( t - \frac{\tau}{2} \right) \right] \right\} \quad (44)$$

$$t \geq \tau$$

### EXPERIMENTAL DETERMINATION OF STRAINS

A rigid 12' x 4' x 2' support tower was erected to carry the pulse generator and the experimental model as shown in Fig. 1. The model was supported from 9' lengths of 0.02" diameter piano wire. The wires were soldered into holes drilled through the bars, at their centroids, at distances from the bar ends corresponding to the location of the nodal points for the fundamental mode of a free-free system. The model tested consisted of two 1/2" x 1/2" x 39.9" parallel bars, elastically connected with 20 coil springs on 2" centers. The spring constant for each spring was 40 pounds per inch. Type A-7, SR-4 strain gages were cemented to each bar at its quarter point and center point using Duco cement. A dummy gage was cemented to a piece of cold rolled stock and placed in the bridge circuit as a temperature compensator. An Ellis BAM-1 Bridge Amplifier Meter was used to amplify strain gage response and it was connected to one beam of a Tektronix Type 502 Dual Beam Oscilloscope to display the strain. A piezoelectric force gage [10] was placed on the bar at the impact point and its output was displayed on the other beam of the oscilloscope. In order to calibrate the piezoelectric crystal, a calibrating device was made from two pieces of 1" x 1" x 1-3/4" bar stock. Three strips of .003" brass shim plate were attached to the 1" x 1" bars to provide a moment-less joint. Weights were placed on a platform which was spliced with solder to the end wire of the calibrating device.

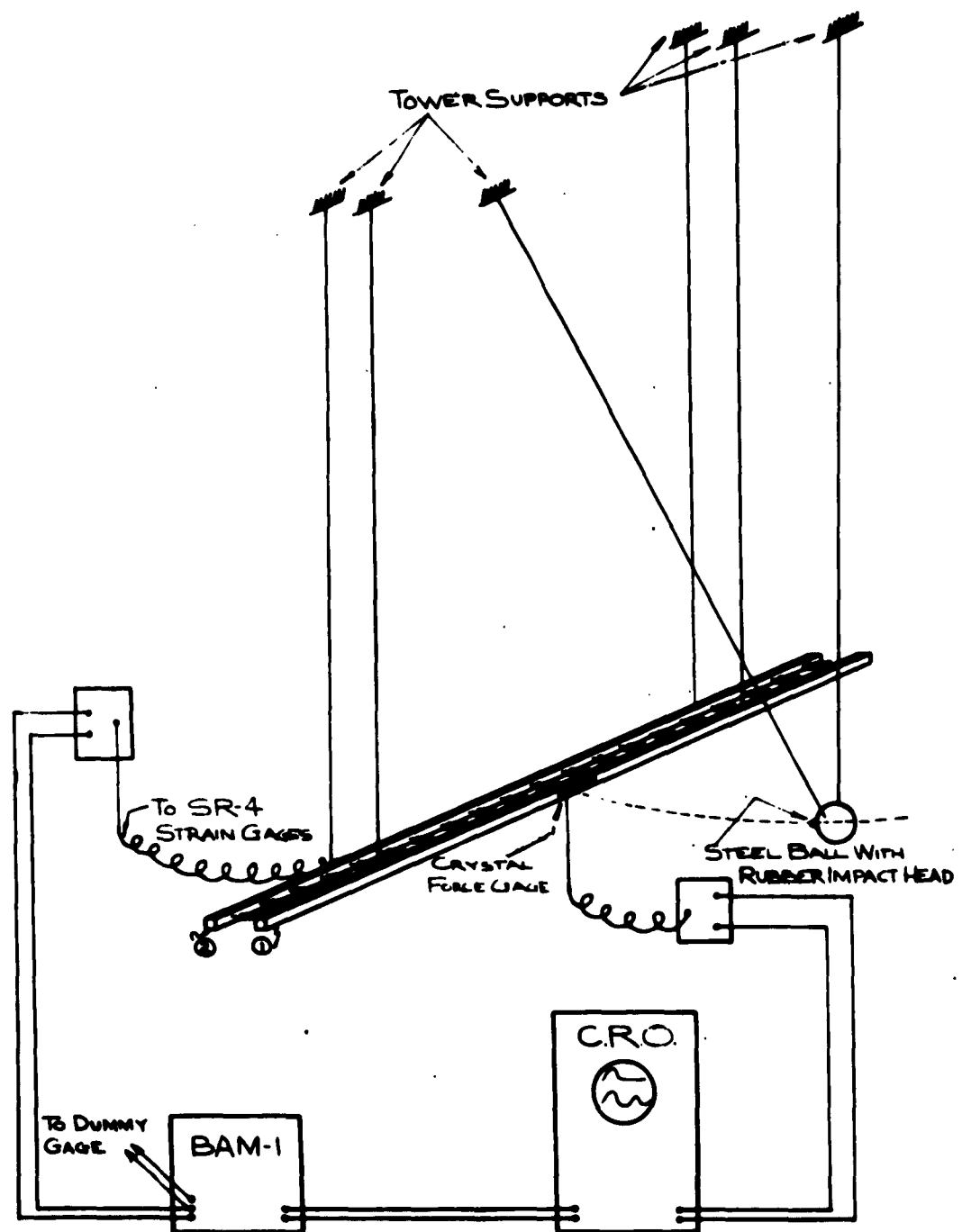


Figure 1. Experimental Apparatus for Impact on  
Elastically Connected Free Double Beams

The distances from the momentless joint to the cap on the crystal and to the end wire were accurately measured. Using these data, the force on the crystal could be determined. The solder was cut using a blowtorch and the load was thereby removed from the crystal. The response of the crystal was displayed on the oscilloscope and photographed using a Tektronix C-12 Polaroid camera. This procedure was followed using different weights and with the results, a calibration curve of force as a function of voltage output was determined. The slope of this curve was .642 pounds/millivolt. The oscilloscope connections from the force gage were shunted by a 1.0 microfarad 200V capacitor. This gave an RC value of 1 second. This RC value varied from about 50 $\tau$  to 100 $\tau$  for the tests run. The response to an approximate half-sine wave impulsive load can accurately be determined with the RC constant between these limits.

As an approximate half-sine wave pulse generator, a 2" diameter steel ball, weighing 1.25 pounds was used. A 1/2" diameter, hard rubber hemisphere, backed by sponge rubber, was cemented to the ball in order to soften the blow imparted to the model and produce a half-sine pulse. The ball was hung with a bifilar suspension of nylon string, from the tower.

The impacting ball was brought back to a predetermined angular position and then allowed to freely swing and strike

the force gage. The oscilloscope circuit was triggered by the force gage response. The oscilloscope sweep speed was known so that the duration of response could be accurately determined. Experiments were performed by impacting the structure both at the quarter point and then at the center point. Strain response was recorded from all the strain gages mounted on the model. For each test the force gage response was displayed and in this manner a check was made to determine that the same impulsive load was applied during each test. The response was quite easily and accurately duplicated.

#### COMPARISON OF EXPERIMENTALLY AND THEORETICALLY DETERMINED STRAINS

The impulsive force used in the theoretical calculation was taken as a half-wave sine approximation to the actual applied force as shown in Fig. 2. Comparisons of the theoretically and experimentally determined strains are shown in Fig. 3 to Fig. 10 inclusive. It may be noted that stress which is assumed to be proportional to strain is what is actually plotted.

#### DISCUSSION AND CONCLUSIONS

The study shows that the Bernoulli-Euler beam theory may be used reliably to determine flexural strains caused in the elastically coupled double beam, at least for a blow soft enough so that not more than about ten modes are perceptibly



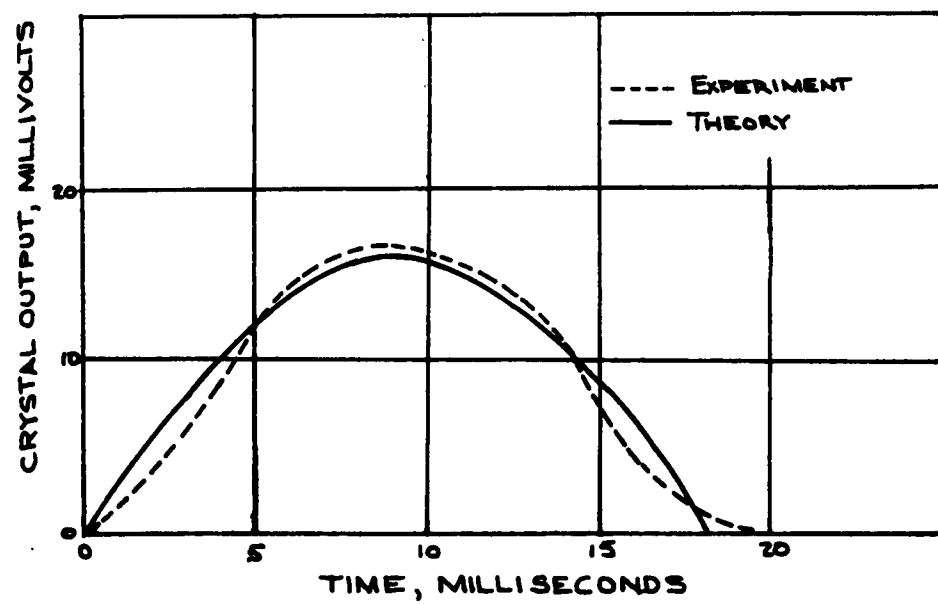


Figure 2. Impact Force

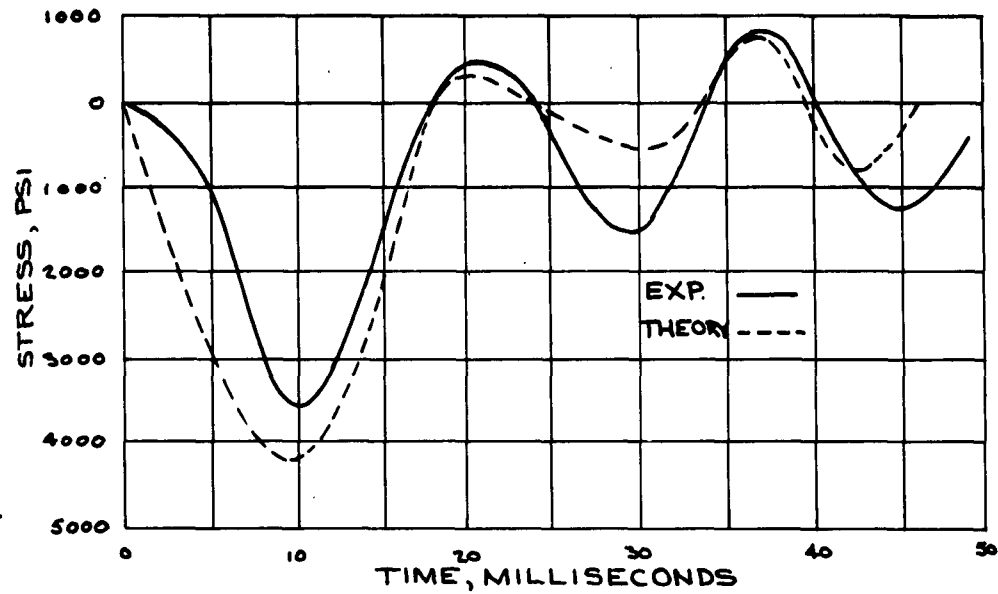


Figure 3. (a) IMPACT AT CENTER LINE, BEAM 1  
STRESS AT CENTER LINE, BEAM 1

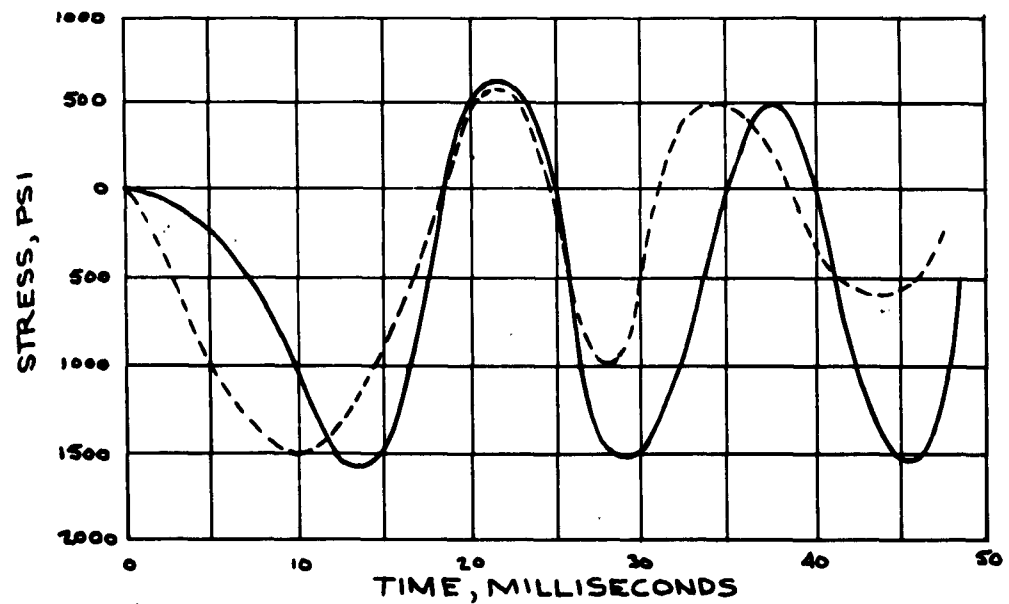


Figure 4. (b) IMPACT AT CENTER LINE, BEAM 1  
STRESS AT CENTER LINE, BEAM 2

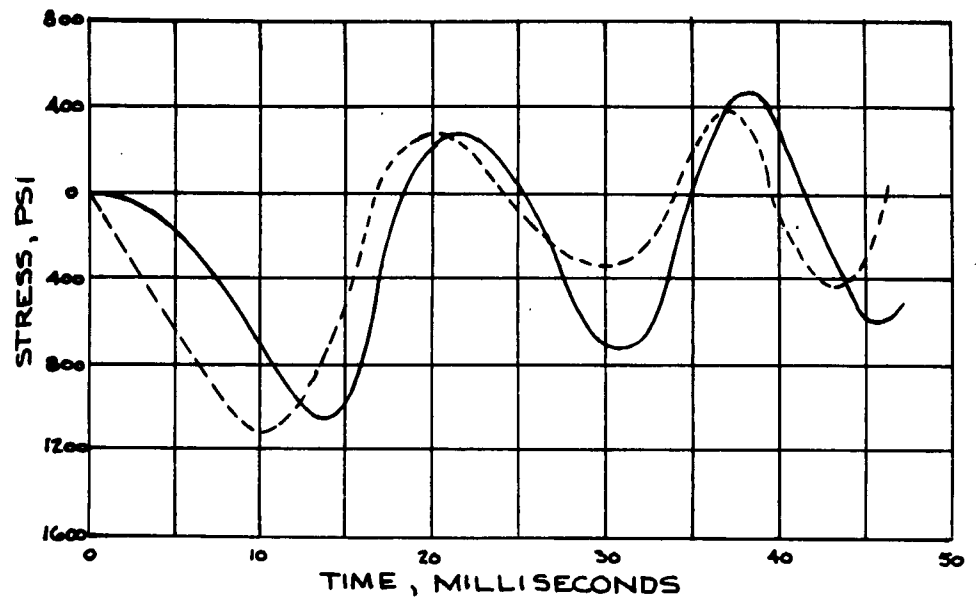


Figure 5. (a) IMPACT AT CENTER LINE, BEAM 1  
STRESS AT QUARTER POINT, BEAM 1

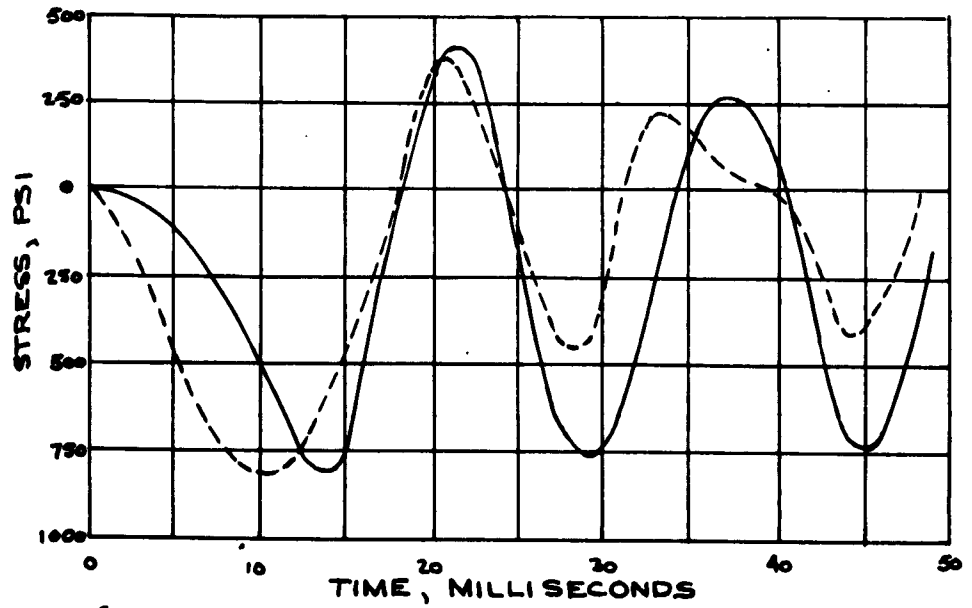


Figure 6. (b) IMPACT AT CENTER LINE, BEAM 1  
STRESS AT QUARTER POINT, BEAM 2

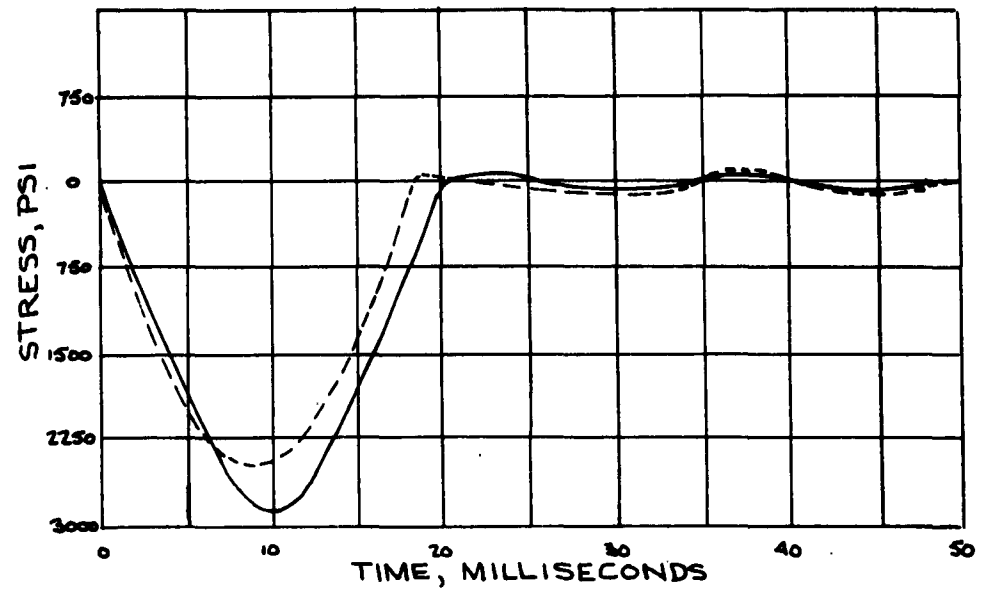


Figure 7. (a) IMPACT AT QUARTER POINT, BEAM 1  
STRESS AT QUARTER POINT, BEAM 1

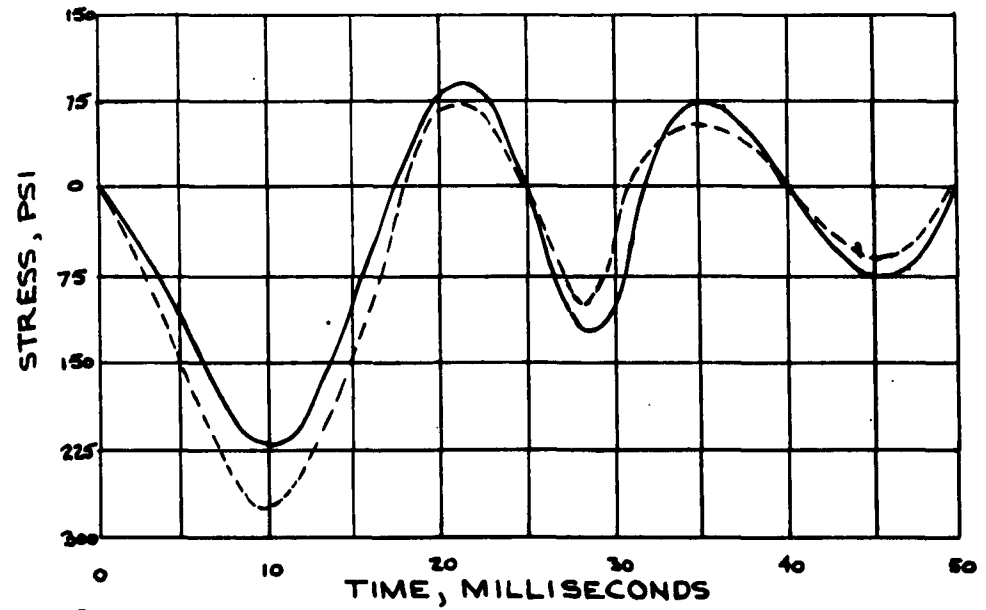


Figure 8. (b) IMPACT AT QUARTER POINT, BEAM 1  
STRESS AT QUARTER POINT, BEAM 2

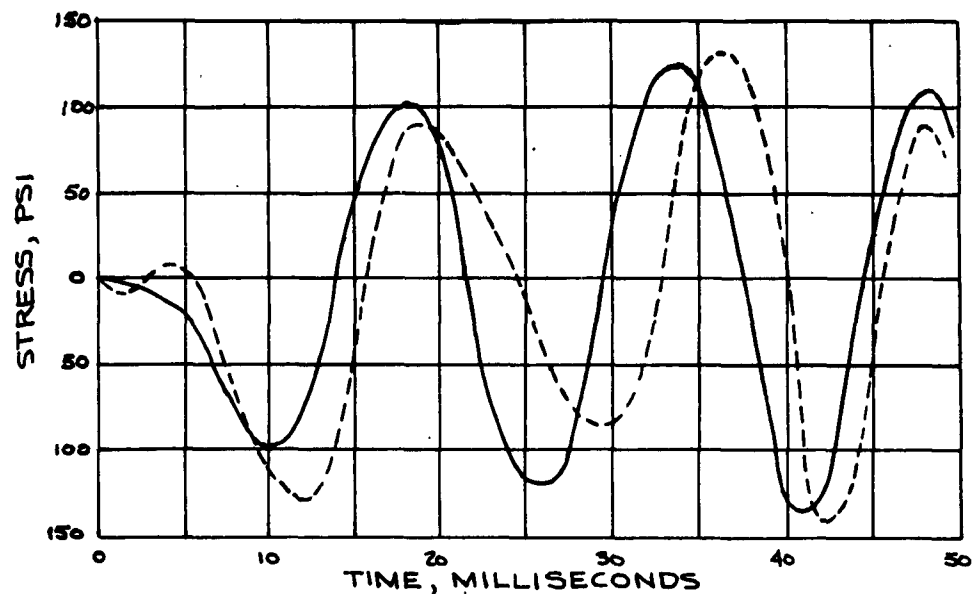


Figure 9. (a) IMPACT AT QUARTER POINT, BEAM 1  
STRESS AT CENTER LINE, BEAM 1

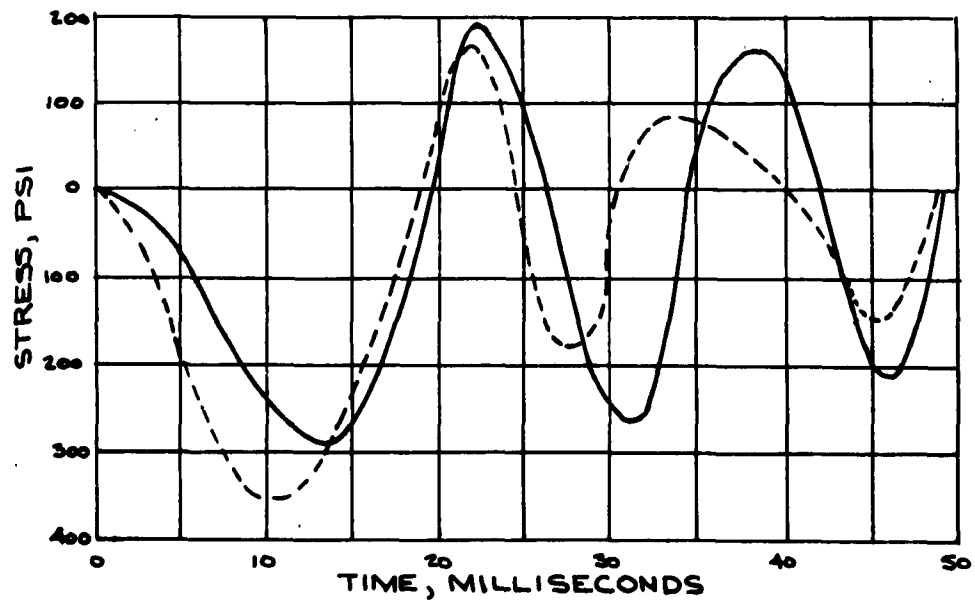


Figure 10. (b) IMPACT AT QUARTER POINT, BEAM 1  
STRESS AT CENTER LINE, BEAM 2

excited. It is considered that the agreement between the measured strains and those predicted by the theory is sufficiently good for many practical design problems.

The load generator developed for the purpose of performing the experiments proved to be very satisfactory. The rubber impacting surfaces applied to the steel striking ball can be fashioned readily to cover a range of hardness from steel-on-steel to very soft rubber on steel.

The load measuring crystal performs excellently in the force ranges encountered in the experiments.

## REFERENCES

1. W. Goldsmith, "Impact", Edward Arnold Publishers, London, 1960, p. 350.
2. C. M. Harris and C. E. Crede, "Shock and Vibration Handbook", McGraw-Hill Book Company, Inc., New York, vol. I, 1961, p. 9-1.
3. H. H. Emschermann and K. Rühl. "Beanspruchungen eines Biegeträgers bei schlagartiger Querbelastrung", VDI-Forschungsheft, 1954, 443.
4. W. H. Hoppmann II, "Impulsive Loads on Beams", Proceedings of the Society for Experimental Stress Analysis, 10, no. 1, 1952, p. 157.
5. M. Hetényi, "Beams on Elastic Foundations", The University of Michigan Press, Ann Arbor, 1946.
6. W. H. Hoppmann II, "Impact of a Mass on a Damped, Elastically Supported Beam", Journal of Applied Mechanics, 15, 1948, p. 125.
7. E. Wenk, "Radial Impact on an Elastically Supported Ring", Transactions of the American Society of Civil Engineers, 119, 1954, p. 221.
8. M. Dublin and H. Friedrich, "Forced Responses of Two Elastic Beams Interconnected by Spring Damper Systems", Journal of the Aeronautical Sciences, 23, 1956, p. 824.
9. E. Osborne, "Computations of Bending Modes and Mode Shapes of Single and Double Beams", Journal of the Society of Industrial and Applied Mathematics, 10, 1962, p. 329.
10. M. Goland, P. D. Wickersham and M. A. Dengler, "Propagation of Elastic Impact in Beams in Bending", Journal of Applied Mechanics, 22, 1955, p. 1.